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Finite-amplitude stability of pipe flow

By A. DAVEY
Department of Applied Mathematics and Theoretical Physics,
University of Cambridge

AND H. P. F. NGUYEN
Computer Science Department, Johns Hopkins University

With an appendix by A. E. GILL, Department of Applied Mathematics
and Theoretical Physics, University of Cambridge

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In this paper we present some results concerning the stability of flow in a circular pipe to small but finite axisymmetric disturbances. The flow is unstable if the amplitude of a disturbance exceeds a critical value, the equilibrium amplitude, which we have calculated for a wide range of wave-numbers and Reynolds numbers. For large values of the Reynolds number, R, and for a real value of the wave-number, α, we indicate that the energy *density* of a critical disturbance is of order c_i^2, where $-\alpha c_i$ is the damping rate of the associated infinitesimal disturbance. The energy, per unit length of the pipe, of a critical disturbance which is concentrated near the axis of the pipe is of order R^{-2}, and the wave-number α is of order $R^{\frac{1}{3}}$. For a critical disturbance which is concentrated near the wall of the pipe the energy is of order $R^{-\frac{3}{2}}$ and α is of order $R^{\frac{1}{2}}$. This suggests that non-linear instability is most likely to be caused by a 'centre' mode rather than by a 'wall' mode. The wall mode solution is also essentially the solution for the problem of plane Couette flow when αR is large. We compare it with the true solution.

In an appendix Dr A. E. Gill indicates how some of the results of this paper may be inferred from a simple scale analysis.

1. Introduction

It is generally believed that the flow of a viscous incompressible fluid in a circular pipe under the action of a constant pressure gradient is stable to both axisymmetric and non-axisymmetric infinitesimal disturbances. This theoretical knowledge is supported by experimental evidence, provided that the experiments are performed with considerable care as regards the inlet conditions and the smoothness of the inner wall of the pipe. However, in the majority of experiments which are executed with only reasonable care, the flow usually becomes unstable when the Reynolds number, based on the pipe radius, exceeds a value of about 2000.

It seems likely therefore that pipe flow is probably unstable if the flow contains small but finite disturbances which are sufficiently large; how large will depend upon their wave-number and the Reynolds number of the flow. In this paper we have calculated these equilibrium amplitudes (which are sometimes referred to in the literature as 'neutral' or 'critical' or 'threshold' amplitudes), using the method of Landau (1944) as developed by Malkus & Veronis (1958), Stuart (1960), Watson (1960) and Reynolds & Potter (1967). The small parameter in the expansion which we shall use, following Watson, is the amplitude A of the disturbance. There is no neutral stability curve for linearized theory in this problem, and we do not pretend that the equilibrium amplitudes which we obtain are particularly accurate. However, we do feel we have shown that the effect of the non-linear terms is to make the flow less stable. Also, our results should give a good indication as to how the size of a critical disturbance will vary with the wave-number α and the Reynolds number R. We discuss the validity of the expansion procedure, and some asymptotic results for large values of αR in §4.

Although it is possible that corresponding, though much more involved, calculations for non-axisymmetric disturbances might yield smaller equilibrium amplitudes, though of the same order of magnitude, we feel nevertheless that the results presented here are important in their own right, and in particular they establish a yardstick for comparison with any future non-linear stability calculations which may be done for the pipe flow problem. The best experimental work on this problem is probably that due to Leite (1959), although his results are not entirely conclusive. In his experiments Leite tried to introduce disturbances with no symmetry. He found, however, that the disturbances became more axisymmetric as they progressed downstream, indicating that the non-axisymmetric part of the disturbance was more heavily damped than the axisymmetric part. This seems to be a good justification for at least considering the problem of axisymmetric disturbances. Some experimental work has also been done by Fox, Lessen & Bhat (1968).

We have considered disturbances which grow, or decay, with time, rather than downstream distance, so as to minimize the amount of computer time required. For comparison with experimental results it would perhaps have been preferable to do otherwise. However, our results for temporally growing, or decaying, disturbances should illustrate all the qualitative features of the problem, and one may use a transformation, due to Gaster (1963), to obtain the corresponding results for spatially amplified disturbances for comparison with experiment.

The principal results of this paper are contained in figures 6, 7 and 8. In figure 6 we indicate how the energy of a critical disturbance which is concentrated near the centre of the pipe depends upon the wave-number and the Reynolds number. In figure 7 we show the corresponding result for a critical disturbance which is concentrated near the wall of the pipe. The solid line drawn in figure 7 is applicable to the problem of plane Couette flow. The true solution for plane Couette flow is given in figure 8.

2. Determination of the Landau constant

We suppose that the fluid is incompressible and has kinematic viscosity ν, that the pipe is infinitely long and that the pressure gradient is maintained at a constant value. The undisturbed flow is parabolic with velocity U_0 on the centre-line. We choose U_0 and a, the pipe radius, as the characteristic velocity and length respectively, with respect to which we make our quantities non-dimensional. Let x, r be the non-dimensional co-ordinates in the streamwise and normal directions respectively. We define a Reynolds number by

$$R = U_0 a/\nu. \tag{1}$$

We shall examine the stability of the flow only to axially symmetric disturbances, so that quantities will be invariant with respect to the azimuthal angle. Let u, v denote the velocities in the x, r directions respectively. An overbar represents the average over one wavelength, and the suffix 'l' refers to the basic laminar flow. The externally applied mean pressure gradient is taken to be invariant as regards a distorted mean flow, so that

$$\overline{p} = -4x/R + \text{constant}. \tag{2}$$

To satisfy the continuity equation we express the velocity field via a stream function ϕ so that

$$u = \frac{1}{r}\frac{\partial \phi}{\partial r}, \quad v = -\frac{1}{r}\frac{\partial \phi}{\partial x}. \tag{3}$$

We consider disturbances which grow with time (temporal) and which are periodic in distance downstream, so that the stream function ϕ may be represented by a Fourier series of the form

$$\phi = \phi_0(r,t) + \phi_1(r,t)\exp(i\chi) + \tilde{\phi}_1(r,t)\exp(-i\chi) + \phi_2(r,t)\exp(2i\chi) + \tilde{\phi}_2(r,t)\exp(-2i\chi) + \text{higher-order terms}, \tag{4}$$

where $\chi \equiv \alpha x + \omega t$. A 'tilde' denotes a complex conjugate and α is a real wave-number.

We shall use the equilibrium amplitude method of Malkus & Veronis (1958) as developed by Reynolds & Potter (1967). Let A be the *real* equilibrium amplitude of a disturbance, then we seek a solution of the Navier–Stokes equations of the form:

$$\phi_1(r,t) = A\psi_1(r) + A^3\psi_{11}(r) + O(A^5), \tag{5}$$

$$\phi_2(r,t) = A^2\psi_2(r) + O(A^4). \tag{6}$$

The mean motion $\overline{u}$ is given by an equation of the form

$$\overline{u} = \frac{1}{r}\frac{\partial \phi_0}{\partial r} = \overline{u}_l + A^2 f_1(r) + O(A^4), \tag{7}$$

where

$$\overline{u}_l = 1 - r^2. \tag{8}$$

The function $f_1(r)$ measures the distortion of the mean motion by the Reynolds stresses.

For our purpose we will not need to know any of the higher-order terms, as we intend to calculate only the first Landau constant Λ_1, so that the frequency ω of the disturbance will be of the form

$$\omega = -\alpha c + \alpha\Lambda_1 A^2 + O(A^4). \tag{9}$$

We must determine the smallest value of A which will make ω real. We choose this method, in preference to that of Watson (1960), which does not appear to be valid for subcritical flows as the distortion of the mean motion cannot be expressed as a power series in the amplitude A. Also, the method of Reynolds & Potter does not require the damping rate of the associated infinitesimal disturbance to be small, moreover, the Landau constants are defined uniquely. Because we only calculate the first Landau constant we hope that the term of order A^2 in (9) will be the dominant non-linear term. Our evaluation of the equilibrium amplitude ignores the other non-linear terms and is therefore only an approximate method; more discussion of this point is given in §4.

The associated infinitesimal disturbance will be of the form $e^{i(\alpha x-\beta t)}$ where α is real and β is complex. If we make the Fourier substitution and the above expansion (9) in powers of A in the Navier–Stokes equations we obtain the following basic equations and boundary conditions.

The equation for the fundamental eigenfunction ψ_1 is

$$L\psi_1 = 0, \quad \text{where} \quad L \equiv [M - i\alpha R(1-r^2-c)]M, \tag{10}$$

and

$$M \equiv D^2 - r^{-1}D - \alpha^2, \quad (D \equiv d/dr). \tag{11}$$

In (10) above $c = \beta/\alpha$ is the wave speed of the disturbance, which is complex, and in (11) we note that the operator $r^{-1}M$ is self-adjoint. The boundary conditions at the centre of the pipe are that the disturbance should be axially symmetric and bounded. Together with the Frobenius series for ψ_1 near $r = 0$ these imply that

$$\psi_1 = r^2 + O(r^4), \quad \text{for small } r, \tag{12}$$

where, for numerical convenience, we take the coefficient of r^2 to be unity as our normalizing condition. We could, however, have chosen many different normalization conditions, and this point will be important in §4 when we discuss the dependence of the terms in the series on the right-hand side of (9), for large values of the parameter αR. The boundary conditions on ψ_1 at the wall are the no-slip conditions

$$\psi_1 = \psi_1' = 0 \quad \text{when} \quad r = 1, \tag{13}$$

where a $'$ denotes differentiation with respect to r. If the flow were inviscid, then the stream function, χ say, satisfies $M\chi = 0$, so that any inviscid solution satisfies the Orr–Sommerfeld equation, though not the boundary conditions.

In order to calculate the integrals which determine the value of the Landau constant Λ_1, we shall need the function θ which is adjoint to ψ_1. We define the adjoint operator $\bar{L}$ and the adjoint function θ, apart from its normalization, so that

$$\int_0^1 \frac{1}{r}\,\theta L\psi\,dr = 0, \tag{14}$$

for any well-behaved function ψ which satisfies the same boundary conditions as ψ_1. Because $r^{-1}[M - i\alpha R(1-r^2-c)]$ is also self-adjoint, we merely have to commute the operators in (10) so that

$$\bar{L}\theta = 0 \quad \text{where} \quad \bar{L} \equiv M[M - i\alpha R(1-r^2-c)]. \tag{15}$$

The boundary conditions are the same as for the eigenfunction ψ_1, namely

$$\left.\begin{array}{ll}\theta = r^2+O(r^4) & \text{for small } r,\\ \theta = \theta' = 0 & \text{when} \quad r = 1.\end{array}\right\} \tag{16}$$

The eigenvalue c obtained from (15) with (16) serves as a check on the numerical work. The normalization of the adjoint function θ does not need to be the same as for ψ_1, we have done this solely for numerical convenience. The vorticity ζ_1 is given by $-r\zeta_1 = M\psi_1$; it follows that $r\zeta_1$ satisfies the adjoint equation $\bar{L}(r\zeta_1) = 0$ and the boundary conditions on θ at $r = 0$ but not those at $r = 1$.

The harmonic function ψ_2 is the dominant term in ϕ_2, the coefficient of e^{2iz}, and we find that for this problem its contribution to the value of the Landau constant is substantial; it derives its energy from the fundamental. The equation for ψ_2 is

$$[N-2i\alpha R(1-r^2-c)]N\psi_2 = \frac{i\alpha R}{r^2}\{(rD\psi_1+2\psi_1)M\psi_1-r\psi_1 DM\psi_1\}, \tag{17}$$

where

$$N \equiv D^2-r^{-1}D-4\alpha^2. \tag{18}$$

The boundary conditions are

$$\psi_2(0) = \psi_2'(0) = \psi_2(1) = \psi_2'(1) = 0. \tag{19}$$

We note that (17) is an inhomogeneous equation for ψ_2 so that we have a direct two-point boundary-value problem. We have to find the coefficients in the power series of ψ_2 for small r which will make $\psi_2(1)$ and $\psi_2'(1)$ zero when we integrate from 0 to 1.

We also need the equation for the function f_1, which, together with A, will tell us by how much the mean flow has been distorted from its parabolic profile. The equation for f_1 is

$$f_1''+\frac{1}{r}f_1' = \frac{i\alpha R}{r^2}\{\tilde{\psi}_1 M\psi_1-\psi_1 M\tilde{\psi}_1\}, \tag{20}$$

and the boundary conditions are

$$f_1'(0) = 0 \quad \text{and} \quad f_1(1) = 0. \tag{21}$$

The formalism of Watson (1960), as used by Ellingsen, Gjevik & Palm (1970) for the problem of plane Couette flow, requires an additional term $-2\alpha Rc_i f_1$ to be added to the left-hand side of (20) where c_i is the imaginary part of the eigenvalue $c = c_r+ic_i$. This term arises from $\partial\bar{u}/\partial t$ which is identically zero in the equilibrium amplitude formulation of Reynolds & Potter, and it is this same term which appears to invalidate the formalism of Watson for subcritical flows. More discussion of this point is contained in § 5. We note that the function f_1, as defined by (20) with (21) is real.

The equation for $\psi_{11}(r)$ is

$$i(\alpha R)^{-1}L\psi_{11} = -\Lambda_1 M\psi_1+g(r), \tag{22}$$

where

$$\begin{aligned} r^2 g(r) \equiv {} & [(r\psi_2'-4\psi_2)M\tilde{\psi}_1+2r\psi_2 DM\tilde{\psi}_1-2(r\tilde{\psi}_1'-\tilde{\psi}_1)N\psi_2-r\tilde{\psi}_1 DN\psi_2] \\ & +r[(rf''-f')\psi_1-rfM\psi_1], \end{aligned} \tag{23}$$

and the boundary conditions on ψ_{11} are the same as those on ψ_1. The operator L is singular and Λ_1 is determined uniquely by multiplying (22) by the adjoint function θ and integrating from $r = 0$ to $r = 1$. Hence the value of Λ_1, the first Landau constant, is given by

$$\Lambda_1 = \int_0^1 r^{-1}\theta g\, dr \Big/ \int_0^1 r^{-1}\theta M\psi_1\, dr. \tag{24}$$

To determine the equilibrium amplitude of a neutral or 'critical' disturbance we do not consider the term in (9) of order A^4. (The probable error involved due to this approximation is discussed in §4.) We require that ω shall be real so that

$$0 = -\alpha c_i + \alpha\Lambda_{1i}A^2, \tag{25}$$

where $\Lambda_1 = \Lambda_{1r} + i\Lambda_{1i}$. It follows that an equilibrium state $A = A_e$, is given by

$$A_e^2 = \frac{c_i}{\Lambda_{1i}}. \tag{26}$$

For the pipe flow problem we know that infinitesimal disturbances are damped so that c_i is negative. Hence, it follows from (26) that Λ_{1i} must be negative for us to obtain a real equilibrium amplitude. This was indeed the case for the range of wave-numbers and Reynolds numbers for which Λ_1 was calculated. Such disturbances are said to be subcritical; the non-linear terms have made the flow less stable. The equilibrium flow will be unstable, in the sense that a disturbance with A just larger than A_e will grow, and if A is just less than A_e it will decay.

3. Numerical methods of solution

The principal numerical difficulty associated with the type of problem discussed in this paper lies in the fact that the characteristic values associated with the Orr–Sommerfeld differential operator L differ greatly in their real parts. The problem is ill-conditioned because the viscous complementary functions dominate the inviscid ones and make it difficult to determine what linear combination of the complementary functions will satisfy the given boundary conditions.

Our calculations cover a range of the parameter αR up to 20,000. For values of αR up to about 3000 one can use straightforward shooting methods with single precision arithmetic on a machine which will store numbers to an accuracy of 12 decimal digits. Using double-precision arithmetic one may extend the value of αR up to about 9000. For values of αR greater than 10,000 one must use special techniques. One method which has been used recently with considerable success is that due to Kaplan (1964), which is discussed in detail by Betchov & Criminale (1967). We found that Kaplan's method worked very well for disturbances which were concentrated near a solid boundary. However, the disturbances with which we are most concerned, in this paper, are concentrated near the centre of the pipe, away from the boundary wall. It was found that for these disturbances Kaplan's method would not work *at all*. Our evidence supports the suggestion by Sharma

(1968), in an important contribution to stability theory, that whatever filter one uses in Kaplan's technique, integration through a viscous region, in generating a well-behaved solution, will be inaccurate. Thus, if one has two viscous regions, one near the boundary wall, and one well away from the wall, then the integration will be inaccurate over the whole range between the two viscous regions. We feel that if one uses Runge–Kutta, or some similar integration technique, it is much better to use a proper orthonormalization procedure. Perhaps the best method for solving problems of this kind is to use the improvement of the orthonormalization technique due to Conte (1966).

For values of αR greater than 9000 we used Chebyshev polynomials and a matrix method, due to Wilkinson (1965), to calculate the eigenvalues; but solely because we had this program readily available. It was, however, rather more inefficient as regards run-time and storage requirements than if we had used an orthonormalization method.

We now describe briefly a very simple method which may be used for the solution of any Orr–Sommerfeld problem however complicated it may be. We developed this method to solve a double boundary-layer problem of sixth-order, partly because Kaplan's method becomes rather difficult to comprehend when one needs multiple filters. Moreover, this method is very easy to understand and to program.

Suppose, for example, that $L(\phi) = 0$ where L is a fourth-order operator containing an eigenvalue c, and suppose that the range of integration is $0 \leqslant x \leqslant 1$, and that we have two boundary conditions on ϕ at each end. Let $\mathbf{y} = \{\phi, \phi', \phi'', \phi'''\}$, and choose say 100 steps of length $h = 0{\cdot}01$. Now if we are given a condition $\mathbf{y} = \mathbf{y}_i$ when $x = ih$ and we integrate to obtain $\mathbf{y} = \mathbf{y}_{i+1}$ at $x = (i+1)h$ then $\mathbf{y}_{i+1} = \mathsf{A}^i\mathbf{y}_i$ where A^i is a 4×4 matrix whose elements will be independent of $\mathbf{y}_i$. By letting $\mathbf{y}_i$ have the values $\{0,0,0,1\}$, $\{0,0,1,0\}$, $\{0,1,0,0\}$ and $\{1,0,0,0\}$ in turn we may readily determine A^i and we can do this for each subinterval to find $\mathsf{A}^0, \mathsf{A}^2, \mathsf{A}^3, \ldots, \mathsf{A}^{99}$.

Now, the important point is that, having decided upon the step-length h and the integration routine to be used, these matrices constitute complete knowledge of the problem. When the problem is not ill-conditioned the relationship $\mathbf{y}_{100} = \mathsf{B}\mathbf{y}_0$ may be obtained by a direct shooting method. For ill-conditioned problems truncation errors and the rapidly growing solutions will *malform* B. But as we have found the A^i separately, we may obtain B from

$$\mathsf{B} = \mathsf{A}^{99}\mathsf{A}^{98}\ldots\mathsf{A}^1\mathsf{A}^0.$$

We may then use any standard iterative technique to find the eigenvalue c and the eigenfunction from $\mathbf{y}_0 = \mathsf{B}^{-1}\mathbf{y}_{100}$. We are guaranteed of a successful calculation because the step-length h may be chosen so small that all the matrices A^i will be well formed, however large αR may be. We must remember to recalculate the A^i each time that we change c in the iteration procedure.

The beauty of the method lies in its simplicity; for an nth order differential system the only difference will be that the matrices A^i will be $n \times n$. There is no need for the integration steps to be of the same length; also, the integration may be over any range and in either direction.

The method may also be used for an inhomogeneous two-point boundary-value problem $L(\phi) = f$. The relationship between $\mathbf{y}_i$ and $\mathbf{y}_{i+1}$ is now of the form

$$\mathbf{y}_{i+1} = \mathsf{A}^i\mathbf{y}_i + \mathbf{f}^i.$$

The $\mathbf{f}^i$ are found by integrating with $\mathbf{y}_i = \{0, 0, 0, 0\}$ and the A^i may then be found as before. The relationship between $\mathbf{y}_0$ and $\mathbf{y}_{100}$ is now given by

$$\mathbf{y}_{100} = \mathsf{B}\mathbf{y}_0 + \mathbf{F},$$

where B is as found before and $\mathbf{F}$ will be a known function of the A^i and $\mathbf{f}^i$.

The method is not as computationally efficient as that of Conte; we feel, however, that the simplicity of *complete* orthonormalization outweighs this disadvantage.

4. Equilibrium amplitude results

Typical eigenvalues of equation (10) are as given in figure 3 of Davey & Drazin (1969). There are two types of disturbances which, on a linear theory, are only slightly damped, and we refer to these as 'centre' modes and 'wall' modes; the names indicate the regions respectively where the critical layers lie. For values of αR between about 3000 and 10,000 the damping rate of the first centre mode is approximately half that of the second centre mode, and the damping rate of the first wall mode is about the same as for the second centre mode. We have concentrated our calculations on the first centre mode, the least damped infinitesimal disturbance. We did also, however, obtain results for the first wall mode (these may be obtained without any difficulty using Kaplan's method), as for large values of αR this is also essentially the solution for the problem of plane Couette flow.

Before we discuss the results in detail we wish to examine the truncation of (9) at the quadratic term when calculating the equilibrium amplitude. We illustrate the idea by first considering the problem for a wall mode, for which the critical layer has a width of order $(\alpha R)^{-\frac{1}{3}}$ for large values of αR. For the moment we also suppose that α is small compared to $R^{\frac{1}{2}}$ so that conditions change most rapidly in the radial direction. It is convenient for this discussion to use a different normalization from that used in the numerical method, namely we require that

$$\psi_1 \sim (\alpha R)^{-\frac{2}{3}}. \qquad (27)$$

Because r is almost unity equation (17) for ψ_2 tells us that

$$(\alpha R)^{-1}D^4\psi_2 \sim D^3\psi_1^2,$$

where the derivative $D \sim (\alpha R)^{\frac{1}{3}}$, so that $\psi_2 \sim (\alpha R)^{-\frac{2}{3}}$. A glance at equation (2.1.12) of Watson (1960) indicates that

$$(\alpha R)^{-1}D^4\psi_n \sim D^3\psi_1\psi_{n-1},$$

so that for all n, $\psi_n \sim (\alpha R)^{-\frac{2}{3}}$. This is why we chose (27) as our normalization condition. Also, we may estimate the order of the mean-motion functions f_n from

$$D^2f_n \sim \alpha R D^2\psi_n^2,$$

so that for all $n, f_n \sim (\alpha R)^{-\frac{1}{3}}$. The first Landau constant Λ_1 is of the same order as terms like $D\psi_2$ and f_1 which are both of order $(\alpha R)^{-\frac{1}{3}}$ so

$$\Lambda_1 \sim (\alpha R)^{-\frac{1}{3}}. \tag{28}$$

Now, if we equate powers of A^4 in (2.1.10) of Watson's paper we see that, if Λ_n denotes the nth Landau constant, then

$$\frac{\Lambda_{n+1}}{\Lambda_n} \sim \frac{\psi_{n+3}}{\psi_{n+1}}, \tag{29}$$

and so all the Λ_n are of order $(\alpha R)^{-\frac{1}{3}}$. (This result was stated by Ellingsen *et al.* (1970) for the plane Couette flow problem.) It follows that the equation to determine A_e^2 obtained by requiring that ω be real in (9) takes the form

$$0 = -c_i + \Lambda_{1i} A^2 \left[1 + \frac{\Lambda_{2i}}{\Lambda_{1i}} A^2 + \frac{\Lambda_{3i}}{\Lambda_{1i}} A^4 + \ldots\right],$$

or

$$0 = -c_i + (\alpha R)^{-\frac{1}{3}} A^2 [a_1 + a_2 A^2 + a_3 A^4 + \ldots], \tag{30}$$

where $a_0, a_1, a_2, \ldots$ are independent of αR for large values of this parameter. But $c_i \sim (\alpha R)^{-\frac{1}{3}}$ and so

$$0 = a_0 + A^2 [a_1 + a_2 A^2 + a_3 A^4 + \ldots]. \tag{31}$$

Thus, as Ellingsen *et al.* discuss, taking $A_e^2 = -a_0/a_1$ will only give a crude approximation to the true equilibrium amplitude. One hopes that the *numerical* convergence of (31) is fairly rapid; the calculation of a_2 is much more involved than that of a_1 and at least the value of A_e obtained from $-a_0/a_1$ should give the correct trend for different values of α and R.

For the centre modes the critical layer has a thickness of order $(\alpha R)^{-\frac{1}{4}}$. For these modes we require, for the moment, that $\alpha \ll R^{\frac{1}{3}}$ and as a normalization condition we impose

$$\psi_1 \sim (\alpha R)^{-1}. \tag{32}$$

Because r is small and of order $(\alpha R)^{-\frac{1}{4}}$ equation (17) for ψ_2 now tells us that

$$D^4\psi_2 \sim \frac{\alpha R}{r^2} D^2 \psi_1^2,$$

where now $D \sim (\alpha R)^{\frac{1}{4}}$, so that $\psi_2 \sim (\alpha R)^{-1}$ and as before we can show that for all n, $\psi_n \sim (\alpha R)^{-1}$. Moreover, (20) tells us that $f_1 \sim \alpha R D^2 \psi_1^2$ so that f_1, and indeed also f_n for all n are of order $(\alpha R)^{-\frac{1}{2}}$. The first Landau constant is of the order of terms like $D^2\psi_2$ and f_1 which are both of order $(\alpha R)^{-\frac{1}{2}}$. Also, (29) is still valid, and as the ψ_n are of the same order, for all n, we have $\Lambda_n \sim (\alpha R)^{-\frac{1}{2}}$. Hence the equation to determine A_e^2 is of the form

$$0 = -c_i + (\alpha R)^{-\frac{1}{2}} A^2 [a_1 + a_2 A^2 + a_3 A^4 + \ldots], \tag{33}$$

where the a_i are independent of αR and we know that $c_i \sim (\alpha R)^{-\frac{1}{2}}$. We have the same situation as before and we hope that the series converges fairly rapidly. Thus our results are approximate but at the very least they do indicate that the non-linear terms make the flow less stable.

We stress that this truncation result is independent of the way in which we normalized ψ_1. For example in §2 we supposed that $\psi_1''(0) = 2$ for numerical convenience. This implies that $\psi_1 \sim (\alpha R)^{-\frac{1}{2}}$ and we may use similar reasoning to show that A will be of order $(\alpha R)^{-\frac{1}{2}}$ and that Λ_n will be of order $(\alpha R)^{n-\frac{1}{2}}$, so that (33) becomes a series in $\alpha R A^2$ instead of A^2. Whatever the normalization on $\psi_1, A\psi_1$ will always be of order $(\alpha R)^{-1}$. A more detailed analysis indicates that (30) and (33) are still valid when α is of order $R^{\frac{1}{2}}$ or $R^{\frac{1}{3}}$ respectively, and we shal return to this point in §5.

In figure 1 we present a typical stream function ψ_1, for $\alpha = 6{\cdot}2$ and $R = 500$ when $c_r = 0{\cdot}9492$ and $c_i = -0{\cdot}0632$ also $\Lambda_{1r} = 2{\cdot}351$ and $\Lambda_{1i} = -23{\cdot}17$. We see

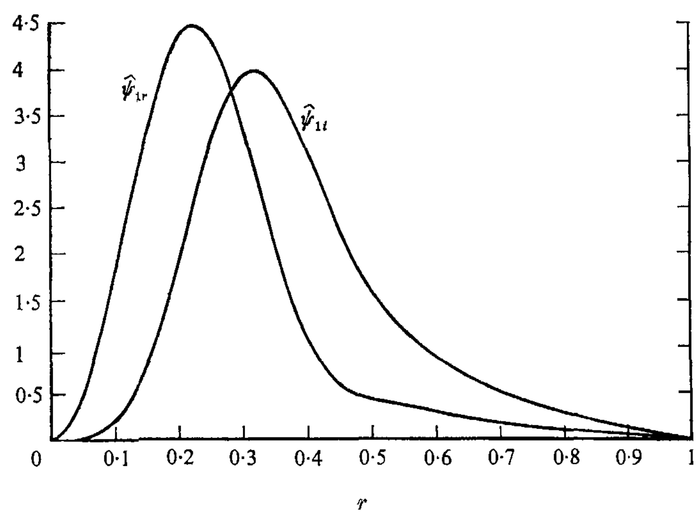

FIGURE 1. The real and imaginary parts of the eigenfunction $\hat{\psi}_1 \equiv R^{\frac{4}{3}} A_e \psi_1$ for $\alpha = \alpha_{\text{crit}} = 6{\cdot}2$ and $R = 500$. The eigenvalue c is given by $c_r = 0{\cdot}9492$ and $c_i = -0{\cdot}0632$.

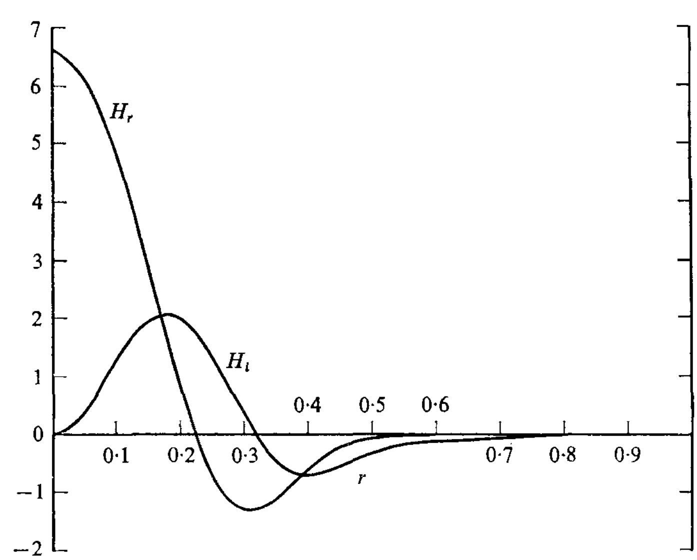

FIGURE 2. The real and imaginary parts of $r^{-1}\psi_1'$ for $\alpha = 6{\cdot}2$ and $R = 500$. We plot $H \equiv R^{\frac{2}{3}} A_e r^{-1} \psi_1'$.

that for $r > 0{\cdot}5$ then ψ_1 is almost a linear function of r and that it vanishes when $r = 1$. In figure 2 we plot $r^{-1}\psi_1'$, which is a measure of the streamwise disturbance velocity. In figure 3 we show the corresponding vorticity ζ_1, which is concentrated near the centre of the pipe and is almost zero for $0{\cdot}4 < r < 1$. We plot $r^{-1}\zeta_1$, which, for inviscid flow, remains constant following a fluid particle. In figure 4 we plot the mean-motion function $f_1(r)$, which determines, apart from the factor A_e^2, the distortion of the mean motion. We note that f_1 is very small outside the critical layer. Also, in figure 4, we plot $G \equiv -A_e^2 f_1'/r$ because $-\bar{u}'/r = 2 + G$ and a necessary condition for instability of an axisymmetric profile $\bar{u}$ is that $|\bar{u}'/r|$ should have a local *maximum* somewhere in the fluid, and G satisfies this condition.

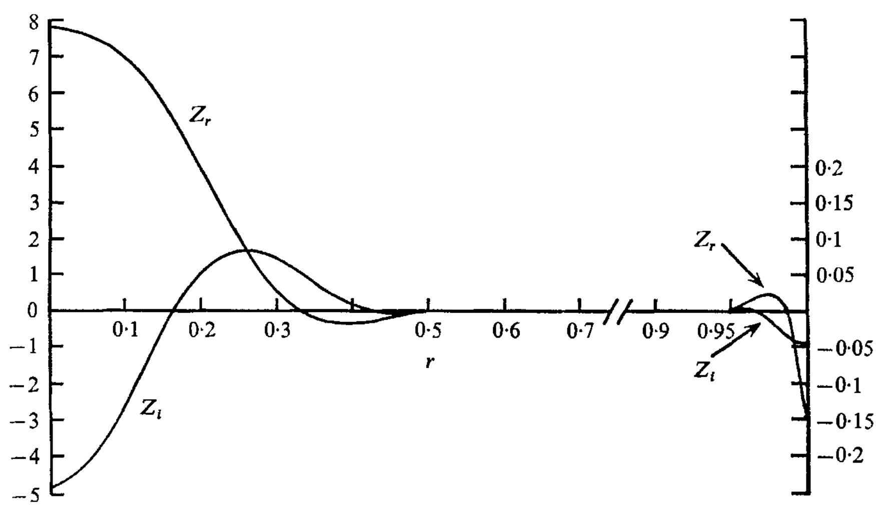

FIGURE 3. The real and imaginary parts of the vorticity ζ_1, divided by r for $\alpha = 6{\cdot}2$ and $R = 500$. We plot $Z \equiv A_e r^{-1}\zeta_1$.

If equation (20) for f_1 had contained the additional term due to $\partial\bar{u}/\partial t$, the character of f_1 would have been changed considerably. We see that f_1 has its maximum value on the axis so that the modified profile contains a jet-like tongue around the axis, and an inflexion point near the axis. The ordinate scaling in figures 1, 2, 3, 4 is chosen so that for values of αR greater than about 2000, the scaled functions will be of the same size for fixed values of $R^{-\frac{1}{3}}\alpha$. We shall see later that, for fixed large R, the value of α of most interest will be the same constant multiple of $R^{\frac{1}{3}}$.

In figure 5 we show some equilibrium amplitude curves which we have obtained for $R = 300$ and $R = 500$ over wave-numbers α from 3 to 8, plotted as curves of constant R. We notice that at a fixed value of the Reynolds number the equilibrium amplitude does not become a minimum until α is quite large. Thus the results for plane Couette flow obtained by Ellingsen *et al.* would have been different if they had done calculations for larger wave-numbers. We indicate in the next section that the important values of α are such that the wavelength of the disturbance is of the *same order* as the thickness of the critical layer.

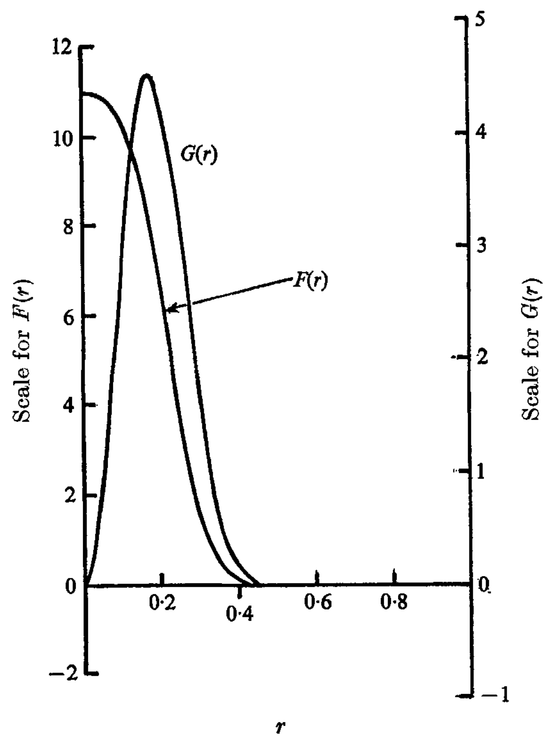

FIGURE 4. The distortion of the mean motion f_1, for $\alpha = 6{\cdot}2$ and $R = 500$. We plot $F \equiv R^{\frac{2}{3}}A_e^2 f_1$, and also $G \equiv -A_e^2 r^{-1} f_1'$.

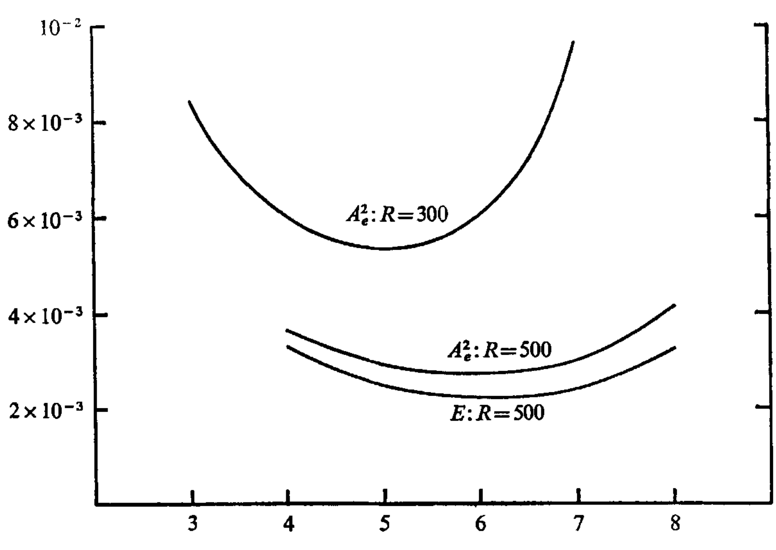

FIGURE 5. The equilibrium amplitude for $R = 300$ and $R = 500$. For comparison we also show the curve of E for $R = 500$.

It is important that the results given in figure 5 should be interpreted correctly. We recall that, for numerical convenience, we normalized our eigenfunction ψ_1 so that $\psi_1 = r^2 + O(r^4)$ for small values of r. This means that on the centre-line of the pipe the axial disturbance velocity is $2A_e U_0$, which will be of order $(\alpha R)^{-\frac{1}{2}} U_0$.

For the whole range of α, R covered we found that the contribution to the value of Λ_{1i} due to the harmonic terms in (23) was smaller than, and opposite in sign to, the contribution due to the mean-motion distortion terms.

5. Discussion

We feel that the question which one would like to be able to answer is as follows: given a value of R, that is a fixed flow speed U_0 on the centre-line for the undisturbed flow, what is the largest disturbance which will still just decay? One could perhaps answer this question if one could determine a value of α which would make, perhaps A_e, a minimum when considered as a function of α. Most calculations in the literature examine the behaviour of A_e as a function of R when α is kept fixed. This does not seem to us to be quite what is required. Our question is also concerned with how one should measure the *size* of a disturbance. Moreover, there seems to be no result in the literature for large values of the wave-number α of the order, say, of R^a for some $a > 0$. For instance, experimental evidence indicates that quite often when a flow becomes unstable, a large wave-number disturbance is initially discernible.

It has been suggested by Professor J. T. Stuart (private communication) that the size, E, of a disturbance may be measured by the ratio of the energy of the disturbance to the energy of the basic flow per unit length of pipe. To first order in A^2, E is given, for the critical disturbance, by

$$E = 12A_e^2 \int_0^1 \{|\psi_1'|^2 + \alpha^2|\psi_1|^2\} \frac{dr}{r}. \tag{34}$$

We suggest that if, in a given problem, and at a *fixed* value of the Reynolds number, E has a minimum for some value of α, then that is what one would like to know. For the centre modes, and for $\alpha \ll R^{\frac{1}{3}}$, the contribution to E comes mainly from the streamwise component of the disturbance velocity so that, for large values of αR, we have $E \sim (\alpha R)^{\frac{1}{2}} A_e^2 \psi_1^2$ and as $A_e^2\psi_1^2 \sim c_i(\alpha R)^{-\frac{1}{2}}$ then $E \sim c_i(\alpha R)^{-1}$. Gill (1963) has shown that, for large values of αR, c_i is given to a very good approximation by

$$-c_i(\alpha R)^{\frac{1}{2}} = (\alpha/R)(\alpha R)^{\frac{1}{2}} + 2^{\frac{3}{2}}. \tag{35}$$

When $\alpha \ll R^{\frac{1}{3}}$ then we see that $E \sim (\alpha R)^{-\frac{3}{2}}$ so that the energy *density* is of order $(\alpha R)^{-1}$ or c_i^2. But this last result must also be true when α is of order $R^{\frac{1}{3}}$, we have simply neglected some terms of the same order of magnitude as those which we have retained. Moreover, in § 6 we will indicate that the energy density is probably of order c_i^2 even when $\alpha \gg R^{\frac{1}{3}}$.

We note that the two terms on the right-hand side of (35) are comparable when $\alpha \sim R^{\frac{1}{3}}$ which is when the wavelength of the disturbance is comparable with the thickness of the critical layer and so we may expect conditions to change. Viscous dissipation due to the radial component of the disturbance velocity becomes important and the second term in the integrand of (34) is of the same order as the first term. For larger values of α we may argue that there will be so much viscous dissipation of energy that E, for a critical disturbance, must increase with α when $\alpha \gg R^{\frac{1}{3}}$. It is probable therefore that E will be a minimum when α is of order $R^{\frac{1}{3}}$† in which case $E_{\min} \sim R^{-2}$. This result is predicted by the scale analysis of the appendix to this paper by Dr A. E. Gill.

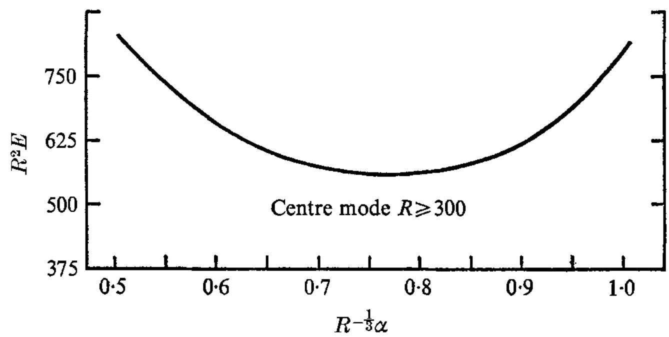

FIGURE 6. The energy E of a critical centre disturbance for $R \geqslant 300$.

This result was also confirmed by our numerical results. In fact we find that the asymptotic theory is valid even when R is as small as 300, the error being exponentially small for large values of αR. In figure 6 we plot R^2E against $R^{-\frac{1}{3}}\alpha$ for the critical disturbance, this curve may be used for all values of $R \geqslant 300$. We see that the critical value of α is approximately $0{\cdot}77R^{\frac{1}{3}}$ and that $E_{\min}$ is approximately $560R^{-2}$.

For the wall modes, and for $\alpha \ll R^{\frac{1}{2}}$, the contribution to E comes again mainly from the horizontal component of the disturbance velocity so that, for large values of αR, we have $E = E_w \sim A_e^2 D\psi_1^2$. For the wall modes $A_e^2\psi_1^2$ is of order $c_i(\alpha R)^{-1}$ and $D \sim (\alpha R)^{\frac{1}{3}}$ so that $E_w \sim c_i(\alpha R)^{-\frac{2}{3}}$. For large values of αR, c_i is given to a very rough approximation by

$$-c_i(\alpha R)^{\frac{1}{3}} = (\alpha/R)(\alpha R)^{\frac{1}{3}} + 1{\cdot}687. \qquad (36)$$

When $\alpha \ll R^{\frac{1}{2}}$ then $E_w \sim (\alpha R)^{-1}$ so that, as for the centre modes, the energy *density* is again of order c_i^2. This result is also true when α is of order $R^{\frac{1}{2}}$. Section 6 and the appendix indicate that this will be true also when $\alpha \gg R^{\frac{1}{2}}$. It follows that for the wall modes E will be a minimum when α is of order $R^{\frac{1}{2}}$ in which case $E_{\min} \sim R^{-\frac{3}{2}}$. Thus $E_{\min}$ will be much larger for the wall modes than for the centre modes, and this is why we concentrate our attention on the centre modes in this paper.

The above asymptotic result was again confirmed by our numerical results. The error in the asymptotic theory is now algebraic for large values of αR. In figure 7 we plot $R^{\frac{3}{2}}E$ against $R^{-\frac{1}{2}}\alpha$ for the critical disturbance, this curve may be used for $R \geqslant 1500$. For comparison the position of the 'dashed' curves are the results for $R = 625$ and $R = 900$. For values of R greater than about 1500 we see

† Thus the most 'dangerous' disturbance will have approximately equal amounts of energy distributed between its different degrees of freedom.

that the critical value of α is approximately $0{\cdot}145R^{\frac{1}{2}}$ and that $E_{\min}$ is approximately $184R^{-\frac{3}{2}}$.

The wall mode solution given by the solid line in figure 7 is, incidentally, the solution for the problem of plane Couette flow with a velocity profile $2U_0y$, and a Reynolds number based on U_0. This is because for large values of αR the wall layer is so thin that the flow does not realise that the pipe is curved, and if $r = 1-y$ for y small it follows that the mean-motion profile near the wall is $2U_0y$.† The true solution for plane Couette flow is given in figure 8.

In table 1 we present a summary of the more important orders of magnitude for both the centre modes and the wall modes.

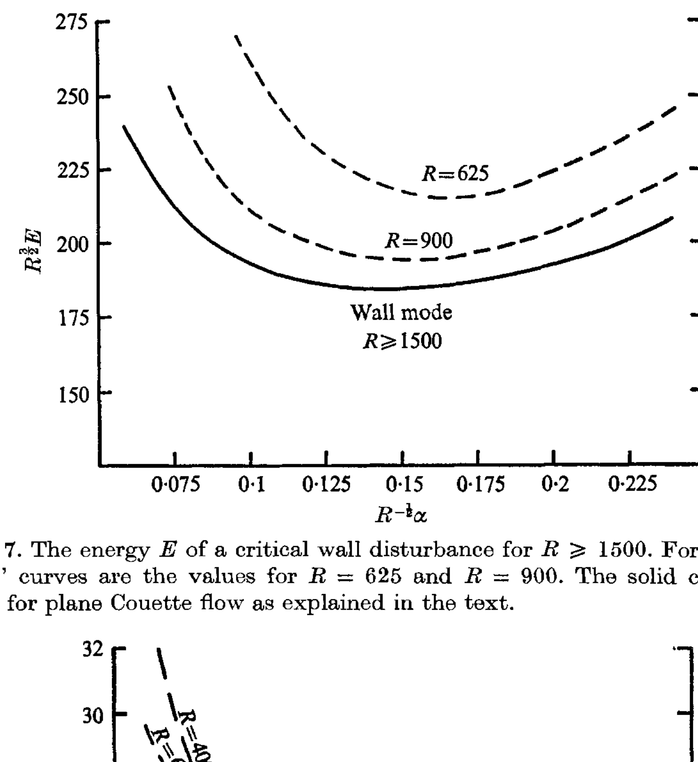

FIGURE 7. The energy E of a critical wall disturbance for $R \geqslant 1500$. For comparison the 'dashed' curves are the values for $R = 625$ and $R = 900$. The solid curve is also the solution for plane Couette flow as explained in the text.

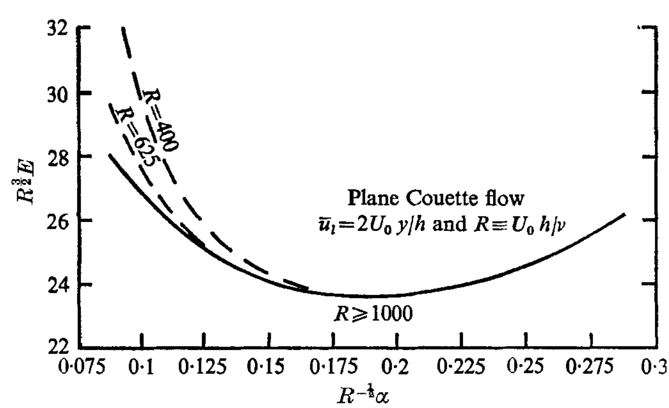

FIGURE 8. The true solution for plane Couette flow, E is as defined by (34) but 12 is replaced by 1·5; $E_{\min}$ is in almost perfect agreement with the value suggested by the wall mode solution.

† The value of E must be divided by 8 to account for the fact that the energy of the basic flow per unit length is different.

It is important to remember that the content of this paper is a *high* Reynolds number theory and the results which we have obtained for, say, $R < 500$ may be only of academic interest. The smaller the value of R, the more important it is to know the values of the higher Landau constants, and to know where the singularities of $\Sigma\Lambda_n z^n$ are, so that an appropriate Eulerian transformation, probably of the form $z^* = z/(1+\gamma z)$, may be used, for some value of $\gamma > 0$.

Mode	$-c_i$	h	α_{crit}	E_{min}	ED	ED_{min}
Centre	$(\alpha R)^{-\frac{1}{2}}$	$(\alpha R)^{-\frac{1}{4}}$	$R^{\frac{1}{3}}$	R^{-2}	c_i^2	$R^{-\frac{4}{3}}$
Wall	$(\alpha R)^{-\frac{1}{3}}$	$(\alpha R)^{-\frac{1}{3}}$	$R^{\frac{1}{2}}$	$R^{-\frac{3}{2}}$	c_i^2	R^{-1}

TABLE 1. Some orders of magnitude. The length scale h is the thickness of the critical layer, and ED is the energy density in this layer

The principal difference between the formalism of Watson (1960) and the equilibrium amplitude approach of Reynolds & Potter (1967) is that Watson's equations take into account the growth (or decay) of the disturbance with time, whereas the equilibrium amplitude analysis supposes that the disturbance has already attained its equilibrium value. A consequence of Watson's method is that an additional term $-2\alpha Rc_i f_1$, appears on the left-hand side of (20), the equation for the distortion of the mean motion. The inclusion of this term, which arises from $\partial\overline{u}/\partial t$, radically alters the solution for f_1 because the associated homogeneous equation for f_1, namely

$$f_1'' + (1/r)f_1' - 2\alpha Rc_i f_1 = 0, \tag{37}$$

with boundary conditions (21) admits eigensolutions when $-2\alpha Rc_i = \gamma^2$ and γ is a zero of the Bessel function J_0. Moreover, the equations for the higher-order functions f_n admit eigensolutions when $-2n\alpha Rc_i = \gamma^2$. Thus, for subcritical flows the expansion of the mean motion $\overline{u}$ as a power series in A^2 does not appear to be valid. The partial differential equation for $\overline{u}$ is similar to the heat conduction equation with a heat source distribution which decays exponentially with time.

We feel that this may have a bearing on the paper by Pekeris & Shkoller (1969). They found, for plane Poiseuille flow, that strong non-linear instability arose from the interaction of a wall mode and a centre mode and a mode of zero frequency (the mean motion), when the frequencies of the wall mode and the centre mode were very close together. They obtained rapidly growing solutions for a 'three-wave' resonance type situation. The resonance arose because the *real* parts of their eigenvalues were not well separated. Their work is an important contribution to our understanding of non-linear stability theory. We feel, however, that their representation of the mean-motion distortion may not have been very accurate. The paper by Hasselmann (1967) is very relevant.

6. Conclusions

The principal result of this paper is that at a fixed large value of the Reynolds number, R, pipe flow will become unstable when the energy density of a small but finite disturbance is of order $R^{-\frac{4}{3}}$ in the critical layer, which will be situated close to the axis of the pipe. The wave-number will be of order $R^{\frac{1}{3}}$, and the disturbance will probably be either axisymmetric or the first non-axisymmetric mode. The results which we present in figure 7 for disturbances which are concentrated near the wall of the pipe are very relevant to the sister problem of plane Couette flow.

It is interesting that for both the centre modes and the wall modes the energy density of a critical disturbance is of order c_i^2. This property invites us to make two conjectures. First, for given α, R the disturbance which will concern us will probably be the one which has the smallest value of $-c_i$. Hence as, for the centre modes, it is believed that the first non-axisymmetric mode gives a smaller value of $-c_i$, this mode may be more important than the first axisymmetric mode, although the order of magnitude for large values of αR will be the same. Second, because the centre modes are far away from the boundary and the wall modes are very close to the boundary, and yet for both cases we obtain the same result, it seems that it should be possible to develop a 'local' theory for finite disturbances in a critical layer, and this is indeed done in the appendix.

If u and ζ are a typical *disturbance* velocity and vorticity in the critical layer for a two-dimensional flow, the equation for the disturbance vorticity ζ is very roughly of the form

$$\zeta_t + u\zeta_x = R^{-1}(\zeta_{xx} + \zeta_{yy}). \tag{38}$$

In (38) a suffix t denotes a time derivative, and suffixes x, y represent derivatives in the streamwise and normal directions respectively. Now a balance between the linear terms in (38) tells us what the damping rate $-\alpha c_i$ of the associated infinitesimal disturbance will be; either ζ_{xx} or ζ_{yy} will dominate on the right-hand side of (38) according to whether $\alpha \gg R^{\frac{1}{2}}$ or $\alpha \ll R^{\frac{1}{2}}$ respectively. Now ζ_t and $u\zeta_x$ will have orders of magnitude $\alpha c_i \zeta$ and $\alpha u \zeta$ respectively and so when $u \sim c_i$ the non-linear terms will be of the same order of magnitude as the linear terms. When α is of order unity, Benney & Bergeron (1969) have examined two-dimensional disturbances for which non-linear effects dominate over Reynolds stresses in the region of the critical layer. Their solutions are neutral disturbances which require $\epsilon(\alpha R)^{\frac{2}{3}} \gg 1$, where ϵ is a measure of the amplitude of the disturbance stream function.

Our work is deficient in that we considered disturbances which were periodic in space rather than time. Nevertheless, we feel that the results presented here embody at least the qualitative features of the pipe flow problem. They should also be useful for comparison with any similar calculations which may be made for non-axisymmetric disturbances which may have smaller damping rates and which may yield smaller equilibrium amplitudes. We hope that our discussion of the equation for the distortion of the mean motion, and the relevant expansion procedure, may stimulate further attention to these points.

We are greatly indebted to Dr D. Schofield for much valuable advice, and to Dr A. E. Gill and Dr S. Richardson for many helpful suggestions .The work of one of us (H.P.F.N) was done whilst he was at the Mathematical Laboratory, University of Cambridge.

Appendix: Some comments on scales

By A. E. GILL, Department of Applied Mathematics and Theoretical Physics, University of Cambridge

In the main paper, it is shown that finite amplitude disturbances to Poiseuille flow in a pipe are damped less rapidly than would be expected from linear theory. The most interesting result is the estimate of the smallest amount of energy an axisymmetric disturbance must have in order to be undamped. The disturbance with this property is, for large Reynolds numbers, confined to a thin region near the centre of the pipe. It follows, therefore, that the characteristic scales of this disturbance can depend only on ν, the kinematic viscosity of the fluid, and $|U_0''|$, the curvature of the velocity profile at the centre of the pipe. The length scale, l_c, of the disturbance is therefore

$$l_c = (\nu/|U_0''|)^{\frac{1}{3}} = a(2R)^{-\frac{1}{3}}, \tag{A 1}$$

and the velocity scale is

$$u_c = (\nu^2|U_0''|)^{\frac{1}{3}} = U_0(2R^{-2})^{\frac{1}{3}}. \tag{A 2}$$

If these scales are used, the large Reynolds number results of the main paper may be expressed in a form which is independent of the Reynolds number. Thus

$$\hat{E}\nu^{-2} = \tfrac{1}{6}\pi E R^2 \tag{A 3}$$

becomes a function of

$$\hat{\alpha} l_c = \alpha(2R)^{-\frac{1}{3}} \tag{A 4}$$

only, as shown in figure 6, $\hat{E}$ being the (dimensional) disturbance energy per unit pipe length and $\hat{\alpha}$ the (dimensional) wave-number. The smallest value of $\hat{E}$ is about $300\nu^2$ when $\hat{\alpha}$ is $0{\cdot}6\, l_c^{-1}$.

The scale analysis requires that the disturbance vorticity/radius be of the same order as the basic-flow vorticity/radius. This quantity is of interest as it is preserved in inviscid axisymmetric motion, and its value for the basic flow is $|U_0''|$ uniformly over the whole of the pipe. For the minimum-energy disturbance, figure 4 shows that the mean vorticity/radius becomes 3 times the basic-flow value at a radius of $1{\cdot}4\, l_c$, and figure 3 shows that the r.m.s. value in the fundamental is 6 times the basic flow value at the centre of the pipe. Thus, if an axisymmetric disturbance is not to decay, it must produce sizeable changes in the vorticity/radius, but it need only make these changes locally near the centre of the pipe. The scales l_c and u_c are, of course, equally appropriate to non-axisymmetric disturbances but no quantitative information is available at present.

Wall modes

One may also have undamped disturbances confined to a thin region near the wall. The scales for these disturbances can only depend on ν and the local shear $|U_w'|$. The length and velocity scales are therefore

$$l_w = (\nu/|U_w'|)^{\frac{1}{2}} = a(2R)^{-\frac{1}{2}}, \tag{A 5}$$

and

$$u_w = (\nu|U_w'|)^{\frac{1}{2}} = U_0(2R^{-1})^{\frac{1}{2}}. \tag{A 6}$$

The area of the region concerned is of order $l_w a$ so that the disturbance energy $\hat{E}$ per unit length of pipe is of order

$$u_w^2 l_w a = a^2 U_0^2 (2R^3)^{-\frac{1}{2}} = \nu^2 (R/2)^{\frac{1}{2}}, \tag{A 7}$$

and so, large compared with that of the undamped centre disturbance. These scales imply that, for large Reynolds numbers,

$$\hat{E}/(u_w^2 l_w a) = \tfrac{1}{6}\pi E (2R^3)^{\frac{1}{2}}, \tag{A 8}$$

is a function only of
$$\hat{\alpha} l_w = \alpha (2R)^{-\frac{1}{2}}, \tag{A 9}$$

as shown in figure 7. The calculations give a value of $\hat{E}$ of $60\, u_w^2 l_w a$ for a wave-number of $0{\cdot}1\, l_w^{-1}$. The vorticity of the disturbed mean flow is twice that of the basic flow at a distance $6 l_w$ from the wall.

Disturbances of given wave-number

A scale analysis is also helpful in discussing disturbances of fixed wave-number. Take, for instance, a disturbance whose dimensional wave-number, $\hat{\alpha} = \alpha a^{-1}$, is small compared with l_c^{-1} so that the downstream scale is large compared with the radial scale. The least-damped infinitesimal disturbance (Pekeris 1948; Gill 1965) has its vorticity confined to a thin region near the centre, and the flow is irrotational outside that region. The vorticity distribution of the disturbance represents a balance between advection, the curvature of the mean flow tending to sharpen gradients, and viscous diffusion which tends to smooth out gradients (Gill 1963). If $\hat{c}$ is the non-dimensional complex wave speed, and l is the radial length scale, this balance implies that

$$\hat{\alpha}(U_0 - \hat{c}) \sim \hat{\alpha} |U_0''| l^2 \sim \nu l^{-2}, \tag{A 10}$$

so that
$$l \sim (\nu/\hat{\alpha}|U_0''|)^{\frac{1}{4}} = l_c (\hat{\alpha} l_c)^{-\frac{1}{4}} = a (2\alpha R)^{-\frac{1}{4}}, \tag{A 11}$$

and
$$U_0 - \hat{c}_r \sim \hat{c}_i \sim (\hat{\alpha}\nu |U_0''|)^{\frac{1}{2}} = u_c (\hat{\alpha} l_c)^{-\frac{1}{2}} = U_0 (\tfrac{1}{2}\alpha R)^{-\frac{1}{2}}. \tag{A 12}$$

Now harmonics generated by self-interaction of such a disturbance will also be irrotational outside a region of radius l. Likewise the Reynolds stress and hence changes in the mean flow will be confined to a region of radius l. For the fundamental to be significantly altered by self-interactions, it is necessary that advection of vorticity by the disturbance be as important as advection by the basic flow. Thus, if u is the disturbance velocity scale, it is required that

$$u \sim U_0 - \hat{c}_r \sim \hat{c}_i \sim u_c (\hat{\alpha} l_c)^{-\frac{1}{2}} = U_0 (\tfrac{1}{2}\alpha R)^{-\frac{1}{2}}. \tag{A 13}$$

It follows that the disturbance vorticity/radius is of order

$$u l^{-2} \sim u_c l_c^{-2} \sim |U_0''|, \tag{A 14}$$

that is, of the same order as the basic flow. This is the same result as was found for the case when $\hat{\alpha} l_c$ is of order unity.

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